Estimation of Out-of-Sample Sharpe Ratio for High Dimensional Portfolio Optimization

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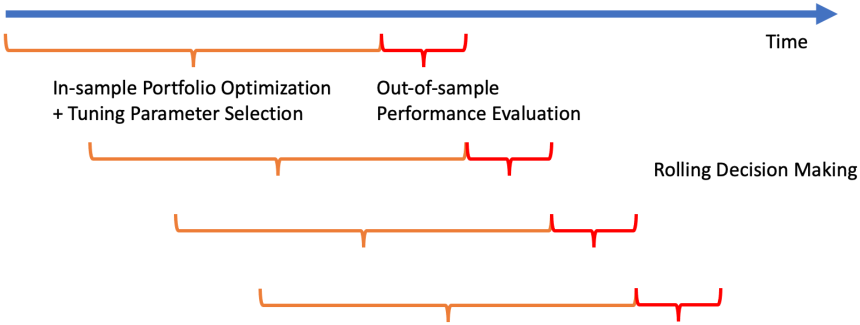
Outline

1. Introduction
2. Estimating OOS Sharpe
3. Extension 1: Estimating Efficient Frontier
4. Extension 2: Optimization Over Regularization
5. Extension 3: Estimating OOS Sharpe with Sample Mean
6. Real Data Experiments
7. Conclusion

Portfolio Management

Portfolio Management



Mean-Variance Portfolio

Definition (Mean-variance portfolio)

Given p risky assets with mean $\mathbf{r} \in \mathbb{R}^p$ and covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{p \times p}$, the mean-variance portfolio optimizes the allocation vector $\mathbf{w}$:

$$\mathbf{w}^* = \operatorname{argmin}_{\mathbf{w} \in \mathbb{R}^p} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \text{s. t.} \quad \mathbf{w}^\top \boldsymbol{\mu} = \mu_0.$$

Here, we denote by $\boldsymbol{\mu} = \mathbf{r} - r_0 \mathbf{1}$ the excess return of the risky assets, and $\mu_0 > 0$ is the targeted excess return of the portfolio.

The solution is $\mathbf{w}^* \propto \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$.
We assume $\boldsymbol{\mu}$ is known and $\boldsymbol{\Sigma}$ needs to be estimated.

Ridge Regularized Mean-Variance Portoflio

ℓ_2-Regularized-MV: Consider the optimization with regularization $\mathbf{Q}$:

$$\mathbf{w}^* = \text{argmin}_{\mathbf{w}\in\mathbb{R}^p}\, \mathbf{w}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})\mathbf{w} \quad \text{s. t.} \quad \mathbf{w}^\top\boldsymbol{\mu} = \mu_0,$$

where $\mathbf{Q}$ is positive definite. The optimal $\mathbf{w}^*$ satisfies

$$\mathbf{w}^* \propto (\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\mu}.$$

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where $\mathbf{Q}$ is positive definite. The optimal $\mathbf{w}^*$ satisfies

$$\mathbf{w}^* \propto (\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\mu}.$$

OOS Sharpe Ratio of $\mathbf{w}^*$:

$$SR(\mathbf{Q}) = \frac{\mathbb{E}_{\widetilde{\mathbf{R}}}[\mathbf{w}^{*\top}(\widetilde{\mathbf{R}} - r_0\mathbf{1})]}{\sqrt{\mathrm{Var}_{\widetilde{\mathbf{R}}}[\mathbf{w}^{*\top}(\widetilde{\mathbf{R}} - r_0\mathbf{1})]}} = \frac{\boldsymbol{\mu}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\mu}}{\sqrt{\boldsymbol{\mu}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\Sigma}(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\mu}}},$$

where $\widetilde{\mathbf{R}}$ is an out-of-sample point with mean $\mathbf{r} = \boldsymbol{\mu} + r_0\mathbf{1}$ and cov $\boldsymbol{\Sigma}$.

Open Problems

★*Is it possible to estimate the out-of-sample Sharpe ratio with some regularization using in-sample data?*

★*Can we then optimize the estimator over the regularization parameter to enhance the out-of-sample Sharpe ratio?*

In-sample Optimism

How about we just use $\widehat{\mathbf{\Sigma}}$ to estimate $\mathbf{\Sigma}$ in $SR(\mathbf{Q})$?

- Assume n iid p-dim return vectors $\boldsymbol{R}_i \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}), i = 1, \ldots, n$, $\boldsymbol{\mu}$ is known and $p/n \to c < 1$.
- The optimized MV portfolio with $\mathbf{Q} = \mathbf{0}$ is based on the sample covariance $\widehat{\mathbf{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{R}_i - \boldsymbol{\mu})(\boldsymbol{R}_i - \boldsymbol{\mu})^\top$, i.e. $\mathbf{w}^* \propto \widehat{\mathbf{\Sigma}}^{-1}\boldsymbol{\mu}$.
- Its in-sample SR is $\sqrt{\boldsymbol{\mu}^\top\widehat{\mathbf{\Sigma}}^{-1}\boldsymbol{\mu}}$ while its out-of-sample SR is $\boldsymbol{\mu}^\top\widehat{\mathbf{\Sigma}}^{-1}\boldsymbol{\mu}/\sqrt{\boldsymbol{\mu}^\top\widehat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma}\widehat{\mathbf{\Sigma}}^{-1}\boldsymbol{\mu}}$.
- Our theorem will show the latter is approximately $(1-c)\sqrt{\boldsymbol{\mu}^\top\widehat{\mathbf{\Sigma}}^{-1}\boldsymbol{\mu}}$, so the in-sample SR is $1/(1-c)$ times larger.
- When c is close to 1, the portfolio performance will be significantly exaggerated.

Estimating OOS Sharpe

Assumptions

1. Observed sample data $\mathbf{R} \in \mathbb{R}^{n\times p}$ satisfies
$$\mathbf{R} = \mathbf{1}_n \mathbf{r}^\top + \mathbf{X},$$
where $\mathbf{X} = \mathbf{Z}\mathbf{\Sigma}^{\frac{1}{2}} \in \mathbb{R}^{n\times p}$. The elements in $\mathbf{Z} \in \mathbb{R}^{n\times p}$ are i.i.d zero mean, variance 1 and finite $(8+\varepsilon)$-order moment for some $\varepsilon > 0$.
2. The portfolio dimension p and the sample size n both tend to infinity, with $p/n \to c > 0$.
3. There exist constants $c_{\mathbf{Q}}, C_{\mathbf{Q}} > 0$ such that $c_{\mathbf{Q}} \leq \lambda_{\min}(\mathbf{Q})$ and $\|\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{Q}\mathbf{\Sigma}^{-\frac{1}{2}}\|_{\text{op}} \leq C_{\mathbf{Q}}$ for any sequences (n, p). In addition, we allow $\mathbf{Q} = \mathbf{0}$ when $c < 1$.

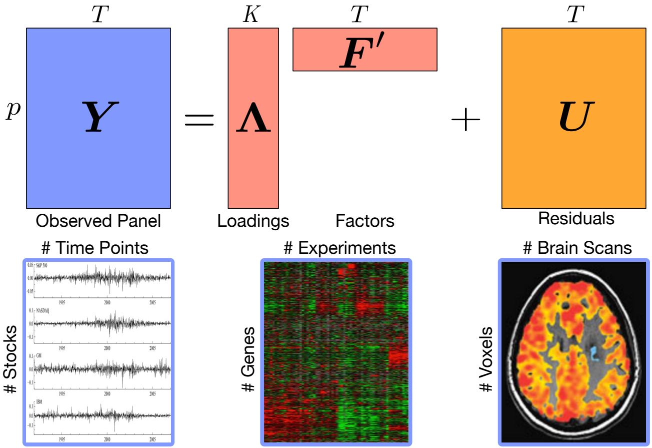
Assumptions

4. $\mathbf{\Sigma}$ is well scaled as $\|\mathbf{\Sigma}/p\|_{\mathrm{tr}} \le C$ for some constant $C > 0$. Denote by $\lambda_1 \ge \cdots \ge \lambda_p$ the eigenvalues of $\mathbf{\Sigma}$. $\lambda_p \ge c_1$ for some constant $c_1 > 0$. One of the following cases must hold:
   - (a) (Bounded spectrum) There exists $C_1 > 0$ such that $\lambda_1 \le C_1$.
   - (b) (Arbitrary number of diverging spikes when $c < 1$) $p/n \to c < 1$ and we allow arbitrary number of top eigenvalues to go to infinity.
   - (c) (Fixed number of diverging spikes when $c \ge 1$) $p/n \to c \ge 1$ and we let the number of diverging spikes be K, K is fixed and $\lambda_1 \le C\lambda_K^2$ for some constant $C > 0$.

Factor models

$$y_{jt} = \boldsymbol{\lambda}_j' \, \mathbf{f}_t + u_{jt}, \quad j = 1, ..., p, \quad t = 1, ..., T \text{ or } n \,, \quad \mathbf{f}_t \in \mathbb{R}^K.$$

Matrix form: $\boldsymbol{Y} = \boldsymbol{\Lambda}\,\mathbf{F}' + \boldsymbol{U}$. $\qquad \mathbb{E}\,\mathbf{f}_t = \mathbf{0}, \;\; \mathbb{E}\, u_{jt} = 0.$



Factor models

$$y_{jt} = \boldsymbol{\lambda}_j^\top \mathbf{f}_t + u_{jt}, \;\; j = 1, ..., p, \;\; t = 1, ..., T \text{ or } n \,, \;\; \mathbf{f}_t \in \mathbb{R}^K.$$

Vector form: $\boldsymbol{y}_t = \boldsymbol{\Lambda}\, \mathbf{f}_t + \boldsymbol{u}_t$.
Covariance structure: $\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{y}_t) = \boldsymbol{\Lambda}\, \mathrm{Cov}(\mathbf{f}_t)\boldsymbol{\Lambda}^\top + \mathrm{Cov}(\boldsymbol{u}_t)$.

- Typically, we assume $\mathrm{Cov}(\boldsymbol{u}_t)$ is a diagonal matrix (strict factor model) and $\mathrm{Cov}(\mathbf{f}_t), \mathrm{Cov}(\boldsymbol{u}_t)$ are well-conditioned.
- For $j \le K$, $\lambda_j(\boldsymbol{\Sigma}) \asymp \lambda_j(\boldsymbol{\Lambda}\, \mathrm{Cov}(\mathbf{f}_t)\boldsymbol{\Lambda}^\top) \asymp \lambda_j(\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top) \asymp \lambda_j(\boldsymbol{\Lambda}^\top\boldsymbol{\Lambda})$ $\asymp \lambda_j(\sum_{j=1}^p \boldsymbol{\lambda}_j\boldsymbol{\lambda}_j^\top) \asymp p$ by law of large numbers.
- For $j > K$, $\lambda_j(\boldsymbol{\Sigma}) \asymp \lambda_j(\mathrm{Cov}(\boldsymbol{u}_t)) \asymp 1$.
- Therefore, we have fixed number K of diverging spikes with $\lambda_j \asymp p$ as $p \to \infty$.

Sharpe Estimation

Define the following quantities:

$$T_{n,1}(\mathbf{Q}) = \operatorname{tr}\left[\left(\frac{\mathbf{X}^\top \mathbf{X}}{n} + \mathbf{Q}\right)^{-1} \mathbf{A}\right],$$

$$T_{n,2}(\mathbf{Q}) = \operatorname{tr}\left[\left(\frac{\mathbf{X}^\top \mathbf{X}}{n} + \mathbf{Q}\right)^{-1} \boldsymbol{\Sigma} \left(\frac{\mathbf{X}^\top \mathbf{X}}{n} + \mathbf{Q}\right)^{-1} \mathbf{A}\right],$$

for some deterministic matrix $\mathbf{A} \in \mathbb{R}^{p \times p}$.

By setting $\mathbf{A} = \boldsymbol{\mu}\boldsymbol{\mu}^\top$, Sharpe Ratio (SR) could be expressed as

$$SR(\mathbf{Q}) = \frac{T_{n,1}(\mathbf{Q})}{\sqrt{|T_{n,2}(\mathbf{Q})|}}.$$

Sharpe Estimation

Theorem

Suppose Assumptions 1-4 hold. For any given $\mathbf{Q}$, *a good estimator* $\widehat{SR}(\mathbf{Q})$ *for* $SR(\mathbf{Q})$ *is as follows.*

$$\widehat{SR}(\mathbf{Q}) = \frac{T_{n,1}(\mathbf{Q})}{\sqrt{\left|\widehat{T}_{n,2}(\mathbf{Q})\right|}}, \quad \textit{where} \quad \widehat{T}_{n,2}(\mathbf{Q}) = \frac{\operatorname{tr}(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\widehat{\mathbf{\Sigma}}(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{A}}{\left(1 - \frac{c}{p}\operatorname{tr}\widehat{\mathbf{\Sigma}}(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\right)^2}.$$

If $\mathbf{A}$ *is semi-positive definite, it holds that*

$$\widehat{SR}(\mathbf{Q})/SR(\mathbf{Q}) \overset{a.s}{\to} 1.$$

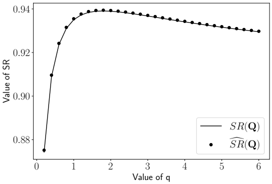
If additionally $\|\mathbf{A}\|_{\mathrm{tr}}$ *is bounded, then* $SR(\mathbf{Q})$ *is almost surely bounded and*

$$\widehat{SR}(\mathbf{Q}) - SR(\mathbf{Q}) \overset{a.s}{\to} 0.$$

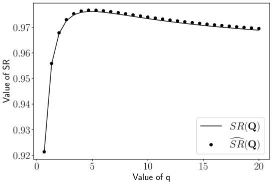
Simulations: Sharpe Estimation

1. Fix $n = 1500$, consider $p = 750$ (ratio $c = 1/2$) and $p = 2250$ (ratio $c = 3/2$).
2. $\mathbf{\Sigma} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) + 2 \cdot \mathbf{1}\mathbf{1}^\top$, where $\{\lambda_i\}_{i=1}^p$ are generated from a truncated $\Gamma^{-1}(1,1)$ distribution, truncated with the interval $[0.01, 9]$, and then ranked in decreasing order.
3. $r_0 = 0$, $\boldsymbol{\mu} = \sqrt{5/p} \cdot (\mathbf{1}(S_+) - \mathbf{1}(S_-)) \in \mathbb{R}^p$. S_+ and S_- are randomly selected subsets of $[p]$ with $|S_+| = |S_-| = p/10$ and $S_+ \cup S_- = \emptyset$.
4. $\mathbf{Q} = q \cdot \mathbf{Q}_0$ where $\mathbf{Q}_0 = \mathrm{diag}(3, ..., 3, 1, ..., 1)$, where the numbers of 3 and 1 entries are both $p/2$. We will vary q.
5. Repeat 1000 times.

Simulations: Sharpe Estimation



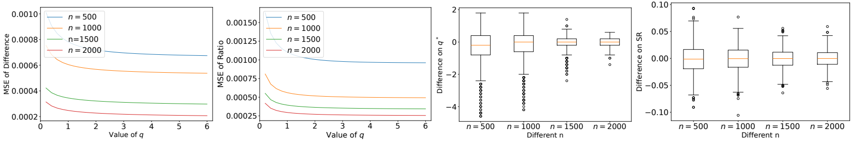
(a) $(n, p) = (1500, 750)$



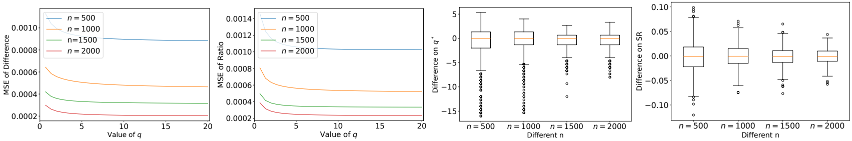
(b) $(n, p) = (1500, 2250)$

Figure 1: Simulation results in the basic settings. Figure 1a shows the case when $c = 1/2$ and Figure 1b depicts the case when $c = 3/2$. The x-axis corresponds to different q values, and the y-axis is the value of SR. The black solid line connects the values of $SR(q \cdot \mathbf{Q}_0)$, while the solid points represent the proposed statistics $\widehat{SR}(q \cdot \mathbf{Q}_0)$.

Simulations: Sharpe Estimation



(a) MSE of diff (b) MSE of ratio (c) q^* diff (d) SR* diff



(e) MSE of diff (f) MSE of ratio (g) q^* diff (h) SR* diff

Figure 2: Simulation results with increasing n. Figures 2a-2d and Figures 2e-2h correspond to $c = 1/2$ and $c = 3/2$. Figures 2a, 2e show $\sum_{b=1}^{1000}(SR_b(q\cdot\mathbf{Q}_0) - \widehat{SR}_b(q\cdot\mathbf{Q}_0))^2/1000$ for different q's. Figures 2b, 2f show $\sum_{b=1}^{1000}(SR_b(q\cdot\mathbf{Q}_0)/\widehat{SR}_b(q\cdot\mathbf{Q}_0) - 1)^2/1000$ for different q's. Figures 2c, 2g give boxplot of $\operatorname{argmax}_q SR_b(q\cdot\mathbf{Q}_0) - \operatorname{argmax}_q \widehat{SR}_b(q\cdot\mathbf{Q}_0)$ for different n's. Figures 2d, 2h give boxplot of $\max_q SR_b(q\cdot\mathbf{Q}_0) - \max_q \widehat{SR}_b(q\cdot\mathbf{Q}_0)$ for different n's.

Estimating Efficient Frontier

Efficient Frontier

When No Risk-free Asset: Given target return $\mu_0 > 0$, the regularized portfolio optimization is given by

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \mathbf{w}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})\mathbf{w}, \quad \text{s.t. } \mathbf{w}^\top\mathbf{r} = \mu_0 \text{ and } \mathbf{w}^\top\mathbf{1} = 1.$$

The optimal $\mathbf{w}^*$ is given by $\mathbf{w}^* = \mathbf{g} + \mu_0 \cdot \mathbf{h}$, where

$$\begin{aligned}
&\mathbf{g} = D^{-1}\Big[B(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1} - A(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r}\Big], \\
&\mathbf{h} = D^{-1}\Big[C(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r} - A(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1}\Big], \\
&A = \mathbf{r}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1},\ B = \mathbf{r}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r}, \\
&C = \mathbf{1}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1},\ D = BC - A^2.
\end{aligned}$$

Efficient Frontier

When No Risk-free Asset: Given target return $\mu_0 > 0$, the regularized portfolio optimization is given by

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \mathbf{w}^\top(\widehat{\mathbf{\Sigma}} + \mathbf{Q})\mathbf{w}, \quad \text{s.t. } \mathbf{w}^\top\mathbf{r} = \mu_0 \text{ and } \mathbf{w}^\top\mathbf{1} = 1.$$

The optimal $\mathbf{w}^*$ is given by $\mathbf{w}^* = \mathbf{g} + \mu_0 \cdot \mathbf{h}$, where

$$\begin{aligned}
&\mathbf{g} = D^{-1}\Big[B(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1} - A(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r}\Big],\\
&\mathbf{h} = D^{-1}\Big[C(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r} - A(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1}\Big],\\
&A = \mathbf{r}^\top(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1},\ B = \mathbf{r}^\top(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{r},\\
&C = \mathbf{1}^\top(\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1}\mathbf{1},\ D = BC - A^2.
\end{aligned}$$

Efficient Frontier: The curve (σ_0, μ_0) as we change target return μ_0, where

$$\sigma_0^2 = \mathbf{w}^{*\top}\mathbf{\Sigma}\mathbf{w}^* = (\mathbf{g} + \mu_0 \cdot \mathbf{h})^\top\mathbf{\Sigma}(\mathbf{g} + \mu_0 \cdot \mathbf{h}),$$

is the variance. Our objective is to estimate σ_0 for any given $\mathbf{Q}$ and μ_0.

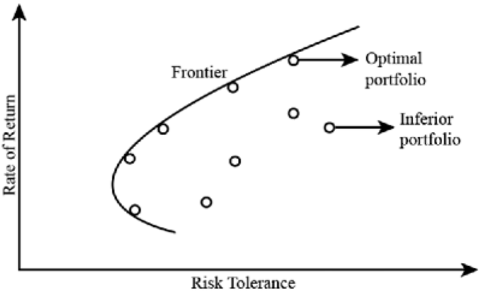
Efficient Frontier

Efficient Frontier with known $\boldsymbol{\Sigma}$:

$$\sigma_0^2 = (\mathbf{g} + \mu_0 \cdot \mathbf{h})^\top \boldsymbol{\Sigma} (\mathbf{g} + \mu_0 \cdot \mathbf{h}) = \mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{g} + 2\mu_0 \mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{h} + \mu_0^2 \mathbf{h}^\top \boldsymbol{\Sigma} \mathbf{h}.$$

When we know true $\boldsymbol{\Sigma}$, we use $\boldsymbol{\Sigma}$ instead of $\widehat{\boldsymbol{\Sigma}} + \mathbf{Q}$ in $\mathbf{g}, \mathbf{h}, A, B, C, D$. Then the above is equivalent to

$$C\sigma_0^2 - C^2/D \cdot (\mu_0 - A/C)^2 = 1. \qquad \text{(Hyperbola)}$$



Assumptions

5. Let $s_0 > 0$ to be the unique solution of the equation.

$$s_0 = \frac{c}{p}\text{tr}\ \boldsymbol{\Sigma}\left(\frac{\boldsymbol{\Sigma}}{1+s_0} + \mathbf{Q}\right)^{-1}.$$

Define

$$\mathcal{A}_{rr} = \mathbf{r}^\top\left(\frac{\boldsymbol{\Sigma}}{1+s_0} + \mathbf{Q}\right)^{-1}\mathbf{r},$$
$$\mathcal{A}_{r1} = \mathbf{r}^\top\left(\frac{\boldsymbol{\Sigma}}{1+s_0} + \mathbf{Q}\right)^{-1}\mathbf{1},$$
$$\mathcal{A}_{11} = \mathbf{1}^\top\left(\frac{\boldsymbol{\Sigma}}{1+s_0} + \mathbf{Q}\right)^{-1}\mathbf{1}.$$

There exists a constant $\rho < 1$ such that $\mathcal{A}_{r1}^2/(\mathcal{A}_{11}\mathcal{A}_{rr}) \leq \rho < 1$.

Efficient Frontier Estimation

Theorem

Suppose that Assumptions 1-5 hold. Define

$$\widehat{\sigma}^2 = \frac{(\mathbf{g} + \mu_0 \mathbf{h})^\top \widehat{\mathbf{\Sigma}} (\mathbf{g} + \mu_0 \mathbf{h})}{(1 - c/p \cdot \mathrm{tr} \widehat{\mathbf{\Sigma}} (\widehat{\mathbf{\Sigma}} + \mathbf{Q})^{-1})^2},$$

where $\mathbf{g}$ *and* $\mathbf{h}$ *are defined as before, it holds that*

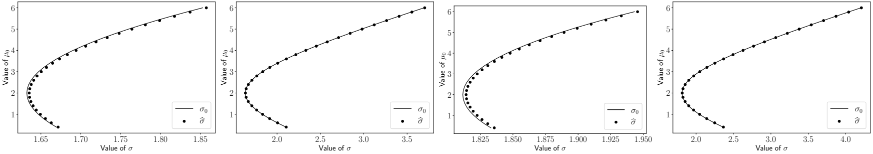
$$\widehat{\sigma}^2 / \sigma_0^2 \stackrel{a.s}{\to} 1.$$

Moreover, the following properties hold:

1. *If* $\mathcal{A}_{rr}$ *is bounded, for any* $r_0 = O(\mu_0)$ *it holds that* $\frac{\mu_0 - r_0}{\sigma_0} - \frac{\mu_0 - r_0}{\widehat{\sigma}} \stackrel{a.s}{\to} 0$.
2. *If* $\mu_0 \leq C\sqrt{\mathcal{A}_{rr}}$ *for some* $C > 0$, *then* $\widehat{\sigma}^2 - \sigma_0^2 \stackrel{a.s}{\to} 0$.

Simulations: Efficient Frontier Estimation

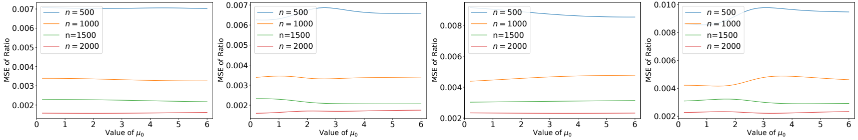
1. Fix $n = 1500$, consider $p = 750$ (ratio $c = 1/2$) and $p = 2250$ (ratio $c = 3/2$).
2. Generate $\boldsymbol{\xi} \in \mathbb{R}^p$ i.i.d. $\Gamma(1,1)$. $\boldsymbol{\Sigma} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) + 2 \cdot \mathbf{1}\mathbf{1}^\top + \boldsymbol{\xi}\boldsymbol{\xi}^\top$. $\boldsymbol{\Sigma}$ respresents a covariance matrix with two factors.
3. $r_0 = 0$, the mean vector $\boldsymbol{\mu}$ here has two choices: $\boldsymbol{\mu} = \boldsymbol{\mu}_1 = p^{\frac{1}{4}} \boldsymbol{\mu}_0 + 2 \cdot \mathbf{1}$ and $\boldsymbol{\mu} = \boldsymbol{\mu}_2 = \boldsymbol{\mu}_0 + 2 \cdot \mathbf{1} + \boldsymbol{\xi}$. $\mathcal{A}_{rr}$ becomes unbounded when $\boldsymbol{\mu} = \boldsymbol{\mu}_1$, while it remains bounded when $\boldsymbol{\mu} = \boldsymbol{\mu}_2$.
4. $\mathbf{Q} = 0.2 \cdot \mathbf{Q}_0$.
5. μ_0 ranges from 0.2 to 6 with the increment of 0.2.
6. Repeat 1000 times.



(a) $\boldsymbol{\mu} = \boldsymbol{\mu}_1, c = 1/2$ (b) $\boldsymbol{\mu} = \boldsymbol{\mu}_2, c = 1/2$ (c) $\boldsymbol{\mu} = \boldsymbol{\mu}_1, c = 3/2$ (d) $\boldsymbol{\mu} = \boldsymbol{\mu}_2, c = 3/2$

Figure 3: Efficient frontiers of $\mathbf{w}^*$. The x-axis is the volatility level, and the y-axis is the target return μ_0. The solid line characterizes the curve of (μ_0, σ_0), while the solid points represent the points in the curve of $(\mu_0, \widehat{\sigma})$.

Simulations: Efficient Frontier Estimation



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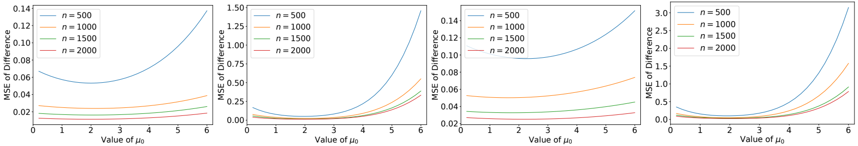


(e) $\boldsymbol{\mu} = \boldsymbol{\mu}_1, c = 1/2$ (f) $\boldsymbol{\mu} = \boldsymbol{\mu}_2, c = 1/2$ (g) $\boldsymbol{\mu} = \boldsymbol{\mu}_1, c = 3/2$ (h) $\boldsymbol{\mu} = \boldsymbol{\mu}_2, c = 3/2$

Figure 4: Simulation results with increasing n. x-axis in all figures shows different values of μ_0. Figures 4a-4d show $\sum_{b=1}^{1000}(\widehat{\sigma}_b^2/\sigma_{0,b}^2 - 1)^2/1000$. Figures 4e-4h show $\sum_{b=1}^{1000}(\widehat{\sigma}_b^2 - \sigma_{0,b}^2)^2/1000$.

Optimization Over Q

Interesting Questions

★***Q1:*** *Given the maximum OOS Sharpe* $SR_{\max} = \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}$*, can* $SR(\mathbf{Q})$ *approach* $SR_{\max}$*? How to predetermine the structure of* $\mathbf{Q}$*?*

★***Q2:*** *Can we optimize* $\mathbf{Q}$ *from* $\widehat{SR}(\mathbf{Q})$*? Define* $\widehat{\mathbf{Q}} = \operatorname{argmax}_{\mathbf{Q}} \widehat{SR}(\mathbf{Q})$*, will the performance of* $SR(\widehat{\mathbf{Q}})$ *be good?*

Answers of Q1

Theorem

Suppose that Assumptions 1-4 hold. Then for any given $\varepsilon > 0$, there exists deterministic sequences of matrices $\widetilde{\mathbf{Q}} \in \mathbb{R}^{p\times p}$ such that with probability 1,

$$1 - \varepsilon \leq \lim_{n\to+\infty} SR(\widetilde{\mathbf{Q}})/SR_{\max} \leq 1.$$

Answers of Q1

Theorem

Suppose that Assumptions 1-4 hold. Then for any given $\varepsilon > 0$, there exists deterministic sequences of matrices $\widetilde{\mathbf{Q}} \in \mathbb{R}^{p\times p}$ such that with probability 1,

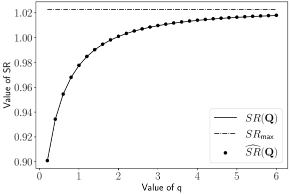
$$1 - \varepsilon \leq \lim_{n\to+\infty} SR(\widetilde{\mathbf{Q}})/SR_{\max} \leq 1.$$

Key of Proof: The existence of $\widetilde{\mathbf{Q}}$ is proved by letting $\widetilde{\mathbf{Q}} = C\boldsymbol{\Sigma}$ for some constant C large enough.

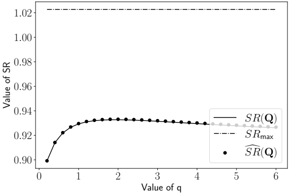
Simulations: Different Structure of $\mathbf{Q}$

1. Fix $n = 1500$, consider $p = 750$ (ratio $c = 1/2$) and $p = 2250$ (ratio $c = 3/2$).
2. $\boldsymbol{\Sigma} = \text{diag}(\lambda_1, \ldots, \lambda_p) + 2 \cdot \mathbf{1}\mathbf{1}^\top$, where $\{\lambda_i\}_{i=1}^p$ are generated from a truncated $\Gamma^{-1}(1, 1)$ distribution, truncated with the interval $[0.01, 9]$, and then ranked in decreasing order.
3. $r_0 = 0$, $\boldsymbol{\mu} = \sqrt{5/p} \cdot (\mathbf{1}(S_+) - \mathbf{1}(S_-)) \in \mathbb{R}^p$. S_+ and S_- are randomly selected subsets of $[p]$ with $|S_+| = |S_-| = p/10$ and $S_+ \cup S_- = \emptyset$.
4. Let $\mathbf{Q}_0 = \text{diag}(3, ..., 3, 1, ..., 1)$, where the numbers of 3 and 1 entries are both $p/2$. Define $\mathbf{Q}_1 = 0.1\mathbf{Q}_0 + q \cdot \text{diag}(\lambda_1, ..., \lambda_p)$; $\mathbf{Q}_2 = 0.5\mathbf{I}_p + q\mathbf{Q}_0$ and $\mathbf{Q}_3 = q\boldsymbol{\Sigma}$. We will vary q.
5. Repeat 1000 times.

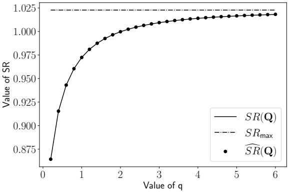
Simulations: Different Structure of $\mathbf{Q}$



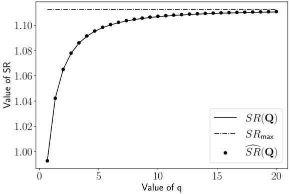
(a) $\mathbf{Q} = \mathbf{Q}_1, c = 1/2$



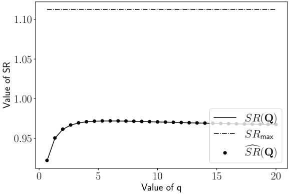
(b) $\mathbf{Q} = \mathbf{Q}_2, c = 1/2$



(c) $\mathbf{Q} = \mathbf{Q}_3, c = 1/2$



(d) $\mathbf{Q} = \mathbf{Q}_1, c = 3/2$



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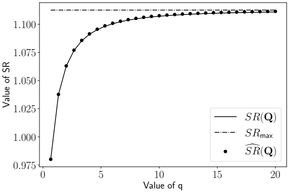

(f) $\mathbf{Q} = \mathbf{Q}_3, c = 3/2$

Figure 5: Simulation results with different $\mathbf{Q}$'s.

Answers of Q2 with Numerical Results

Table 1: Comparison for $SR(\widehat{\mathbf{Q}})$ and $\widehat{SR}(\widehat{\mathbf{Q}})$. The mean gives the average value and the range gives the minimum and maximum values over the 20 independent trials.

(n,p)	$SR_{\max}$	Optimization over full $\mathbf{Q}$			
		mean of $SR(\widehat{\mathbf{Q}})$	Range of $SR(\widehat{\mathbf{Q}})$	mean of $\widehat{SR}(\widehat{\mathbf{Q}})$	Range of $\widehat{SR}(\widehat{\mathbf{Q}})$
$(500, 250)$	0.923	0.643	[0.604,0.694]	1.299	[1.178,1.454]
$(1000, 500)$	1.123	0.791	[0.738,0.824]	1.513	[1.406,1.578]
(n,p)	$SR_{\max}$	Optimization over diagonal $\mathbf{Q}$			
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$(500, 250)$	0.923	0.770	[0.715,0.818]	0.967	[0.909,1.056]
$(1000, 500)$	1.123	0.944	[0.912,0.979]	1.146	[1.081,1.224]

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Primary Reason: $\widehat{\mathbf{Q}}$ is overfitted to in-sample data, breaking down the independence between $\widehat{\boldsymbol{\Sigma}}$ and $\mathbf{Q}$.

Assumptions

6. There exists universal constants $l, L > 0$ such that for all $\mathbf{Q} \in \mathcal{Q}$, both $SR(\mathbf{Q})$ and $\widehat{SR}(\mathbf{Q})$ satisfy $l \leq SR(\mathbf{Q}),\ \widehat{SR}(\mathbf{Q}) \leq L$ almost surely for all n large enough.
7. There exists a sequence of bijections $\phi_n : \mathcal{B} \to \mathcal{Q}$, where $\mathcal{B} \subset \mathbb{R}^k$ is a fixed compact set (independent of n) for some constant $k > 0$. Furthermore, the sequence $\{\phi_n\}$ is equicontinuous with respect to the operator norm: for any $\varepsilon > 0$, there exists $\delta > 0$ (independent of n) such that for all n and all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{B}$,

$$\|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\|_2 \leq \delta \quad \Longrightarrow \quad \|\phi_n(\boldsymbol{\alpha}) - \phi_n(\boldsymbol{\alpha}')\|_{\text{op}} \leq \varepsilon.$$

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Remark: $\mathcal{Q}$ is the candidate set for $\mathbf{Q}$. A specific example for $\mathcal{Q}$ is $\phi_n(\boldsymbol{\alpha}) = \alpha_1 \mathbf{Q}_1 + \alpha_2 \mathbf{Q}_2 + \cdots + \alpha_k \mathbf{Q}_k$ where $\mathbf{Q}_1, \ldots, \mathbf{Q}_k$ are predetermined matrices with $\|\mathbf{Q}_j\|_{\mathrm{op}}$ bounded for all j, $\mathbf{Q}_j$ are linearly independent, and the coefficients $\boldsymbol{\alpha}$ vary over a compact set in $\mathbb{R}^k$.

Consistency Results for Optimal $\mathbf{Q}$

Theorem

Suppose that Assumptions 1-4 and 6-7 hold. Define

$$\widehat{\mathbf{Q}} = \operatorname{argmax}_{\mathbf{Q} \in \mathcal{Q}} \widehat{SR}(\mathbf{Q}).$$

It holds that

$$\widehat{SR}(\widehat{\mathbf{Q}})/SR(\widehat{\mathbf{Q}}) \overset{a.s}{\to} 1, \quad \widehat{SR}(\widehat{\mathbf{Q}}) - SR(\widehat{\mathbf{Q}}) \overset{a.s}{\to} 0.$$

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Control of Overfitting Issues: When the search space for $\mathbf{Q}$ ($\mathcal{Q}$) is well behaved and restricted to a finite dimensional family, the overfitting issue can be controlled, and the optimized candidate achieves consistent performance in the large-sample limit.

Estimating with Sample Mean $\widehat{\boldsymbol{\mu}}$

OOS Sharpe with Sample Mean $\widehat{\boldsymbol{\mu}}$

ℓ_2-Regularized-MV: Consider the optimization with regularization $\mathbf{Q}$:

$$\mathbf{w}^* = \operatorname{argmin}_{\mathbf{w} \in \mathbb{R}^p} \mathbf{w}^\top (\widehat{\boldsymbol{\Sigma}} + \mathbf{Q}) \mathbf{w} \quad \text{s. t.} \quad \mathbf{w}^\top \widehat{\boldsymbol{\mu}} = \mu_0,$$

where $\mathbf{Q}$ is positive definite. The optimal $\mathbf{w}^*$ satisfies

$$\mathbf{w}^* \propto (\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1} \widehat{\boldsymbol{\mu}}.$$

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$$\mathbf{w}^* \propto (\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\widehat{\boldsymbol{\mu}}.$$

OOS Sharpe Ratio of $\mathbf{w}^*$:

$$SR(\mathbf{Q}) = \frac{\mathbb{E}_{\widetilde{\mathbf{R}}}[\mathbf{w}^{*\top}(\widetilde{\mathbf{R}} - r_0\mathbf{1})]}{\sqrt{\operatorname{Var}_{\widetilde{\mathbf{R}}}[\mathbf{w}^{*\top}(\widetilde{\mathbf{R}} - r_0\mathbf{1})]}} = \frac{\widehat{\boldsymbol{\mu}}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\mu}}{\sqrt{\widehat{\boldsymbol{\mu}}^\top(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\boldsymbol{\Sigma}(\widehat{\boldsymbol{\Sigma}} + \mathbf{Q})^{-1}\widehat{\boldsymbol{\mu}}}},$$

where $\widetilde{\mathbf{R}}$ is an out-of-sample point with mean $\mathbf{r} = \boldsymbol{\mu} + r_0\mathbf{1}$ and cov $\boldsymbol{\Sigma}$.

Assumptions

8. Observed sample data $\mathbf{R} \in \mathbb{R}^{n \times p}$ satisfies

$$\mathbf{R} = \mathbf{1}_n \mathbf{r}^\top + \mathbf{X},$$

where $\mathbf{X} = \mathbf{Z}\boldsymbol{\Sigma}^{\frac{1}{2}} \in \mathbb{R}^{n \times p}$. The elements in $\mathbf{Z} \in \mathbb{R}^{n \times p}$ are i.i.d zero mean, variance 1 Gaussian random variables.

Sharpe Estimation with $\widehat{\boldsymbol{\mu}}$

Theorem

Suppose Assumptions 2-4 and 8 hold. For any given $\mathbf{Q}$, *a good estimator* $\widehat{SR}(\mathbf{Q})$ *for* $SR(\mathbf{Q})$ *is as follows.*

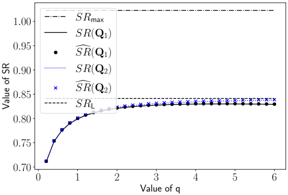
$$\widehat{SR}(\mathbf{Q}) = \frac{\widehat{\boldsymbol{\mu}}^\top(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\widehat{\boldsymbol{\mu}} - \frac{\mathrm{tr}(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\widehat{\boldsymbol{\Sigma}}}{n-\mathrm{tr}(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\widehat{\boldsymbol{\Sigma}}}}{\sqrt{\widehat{\boldsymbol{\mu}}^\top(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\widehat{\boldsymbol{\Sigma}}(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\widehat{\boldsymbol{\mu}}}} \cdot \left(1 - \frac{c}{p}\mathrm{tr}\widehat{\boldsymbol{\Sigma}}(\widehat{\boldsymbol{\Sigma}}+\mathbf{Q})^{-1}\right).$$

If $\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}\|_2$ *is bounded and* $\boldsymbol{\mu}^\top(\frac{\boldsymbol{\Sigma}}{1+s_0}+\mathbf{Q})^{-1}\boldsymbol{\mu}$ *is lower bounded by some constant, it holds that*

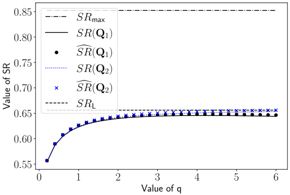
$$\widehat{SR}(\mathbf{Q})/SR(\mathbf{Q}) \overset{a.s}{\to} 1.$$

Simulations: Sharpe Estimation with $\widehat{\boldsymbol{\mu}}$

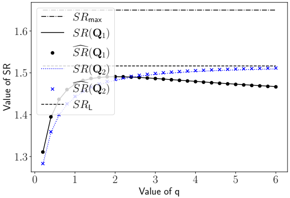
1. Fix $n = 1500$, consider $p = 750$ (ratio $c = 1/2$) and $p = 2250$ (ratio $c = 3/2$).
2. $\boldsymbol{\Sigma} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) + 2 \cdot \mathbf{1}\mathbf{1}^\top$, where $\{\lambda_i\}_{i=1}^p$ are generated from a truncated $\Gamma^{-1}(1, 1)$ distribution, truncated with the interval $[0.01, 9]$, and then ranked in decreasing order.
3. $r_0 = 0$, $\boldsymbol{\mu}_0 = \sqrt{5/p} \cdot (\mathbf{1}(S_+) - \mathbf{1}(S_-)) \in \mathbb{R}^p$. S_+ and S_- are randomly selected subsets of $[p]$ with $|S_+| = |S_-| = p/10$ and $S_+ \cup S_- = \emptyset$. For $\boldsymbol{\mu}_3$, we assume that each element follows an independent uniform distribution, $\mathrm{Unif}(-\sqrt{2/p}, \sqrt{2/p})$, $\boldsymbol{\mu}_4 = \boldsymbol{\mu}_3 + 2 \cdot \mathbf{1}_p$.
4. $\mathbf{Q}_1 = q \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\mathbf{Q}_2 = q\boldsymbol{\Sigma}$. We will vary q.
5. Repeat 1000 times.



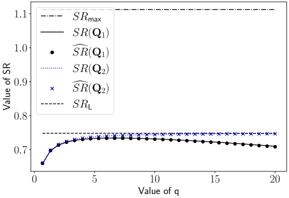
(a) $\boldsymbol{\mu} = \boldsymbol{\mu}_0, c = 1/2$



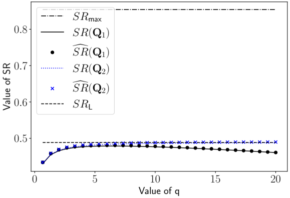
(b) $\boldsymbol{\mu} = \boldsymbol{\mu}_3, c = 1/2$



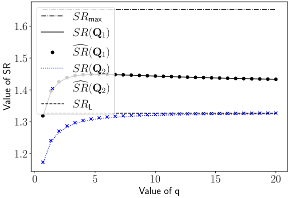
(c) $\boldsymbol{\mu} = \boldsymbol{\mu}_4, c = 1/2$



(d) $\boldsymbol{\mu} = \boldsymbol{\mu}_0, c = 3/2$



(e) $\boldsymbol{\mu} = \boldsymbol{\mu}_3, c = 3/2$



(f) $\boldsymbol{\mu} = \boldsymbol{\mu}_4, c = 3/2$

Figure 6: $SR_{\max} = \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}$, and $SR_L = SR^2_{\max} / \sqrt{SR^2_{\max} + c}$.

Real Data Experiments

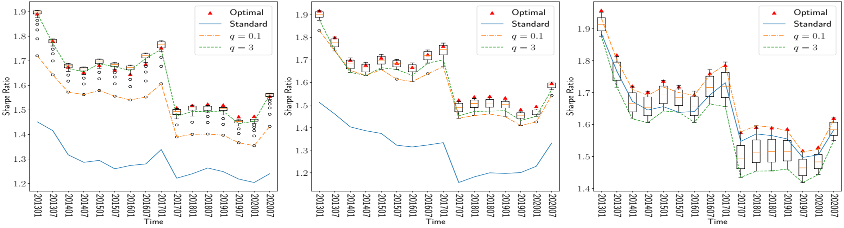
Real Data: Mean-Variance Portfolio

- After deleting stocks with missing values, we have $p = 365$ stocks.
- Portfolios are built using historical data spanning 1, 2, 4 years, and rebalanced monthly.
- Each allocation vector $\mathbf{w}^*$ is held for the entire future testing month. We then have returns of the portfolio $\mathbf{w}^*$ in each trading day of the month.
- For now, we use OOS average return as $\boldsymbol{\mu}$ in each testing month for portfolio construction (known $\boldsymbol{\mu}$).
- Repeat the procedure in a rolling fashion for all testing months from Jan 2013 to Jun 2023 and record daily returns for each trading day.

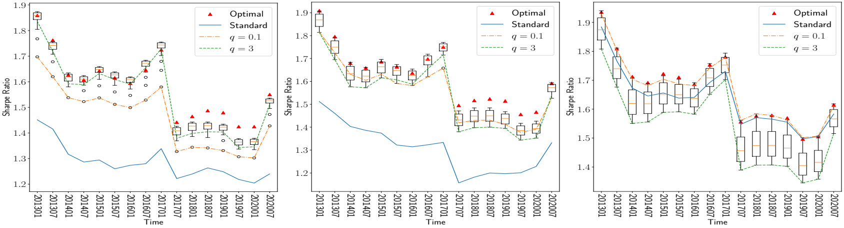
Real Data: Mean-Variance Portfolio

1. Consider two candidate sets. $\mathcal{Q}_1 = \{q \cdot \widehat{\boldsymbol{\Sigma}}_{pre}, q \in [1:30]/10\}$, where $\widehat{\boldsymbol{\Sigma}}_{pre}$ represents the sample covariance pre-trained from 2004 to 2008, not overlapping with data for portfolio construction and evaluation. $\mathcal{Q}_2 = \{q \cdot \mathbf{I}_p, q \in [1:30]/10\}$, where $\mathbf{I}_p$ is the identity matrix.
2. Calculate sample cov $\widehat{\boldsymbol{\Sigma}}$ with 1,2,4-year historical data and construct the regularized MV portfolio.
3. For each testing month, we run experiments for all candidate q values and also consider no regularization, i.e. $q = 0$, where we have $\mathbf{w} \propto \widehat{\boldsymbol{\Sigma}}^+ \boldsymbol{\mu}$ and $\widehat{\boldsymbol{\Sigma}}^+$ is the pseudo inverse, and the optimized $q^* \in \mathcal{Q}$ using our estimator. Note that q^* changes from month to month.
4. We report the average Sharpe ratio of daily portfolio returns over the future three years.

Real Data: Mean-Variance Portfolio



(a) 1 year, $\mathcal{Q} = \mathcal{Q}_1, c > 1$ (b) 2 years, $\mathcal{Q} = \mathcal{Q}_1, c < 1$ (c) 4 years, $\mathcal{Q} = \mathcal{Q}_1, c < 1$



(d) 1 year, $\mathcal{Q} = \mathcal{Q}_2, c > 1$ (e) 2 years, $\mathcal{Q} = \mathcal{Q}_2, c < 1$ (f) 4 years, $\mathcal{Q} = \mathcal{Q}_2, c < 1$

Figure 7: SR of mean-variance portfolios. The x-axis labels the rolling period, while the

Real Data: Global Minimum Variance Portfolio

Using future average returns as $\boldsymbol{\mu}$ is not feasible in practical portfolio construction. One remedy approach is:

- Consider GMV portfolio, which does not require the knowledge of $\boldsymbol{\mu}$:

$$\mathbf{w}^* = \frac{(\widehat{\boldsymbol{\Sigma}} + q\mathbf{I})^{-1}\mathbf{1}}{\mathbf{1}^\top(\widehat{\boldsymbol{\Sigma}} + q\mathbf{I})^{-1}\mathbf{1}}.$$

  Then we check which GMV portfolio attains the minimum OOS empirical variance.

Real Data: Global Minimum Variance Portfolio

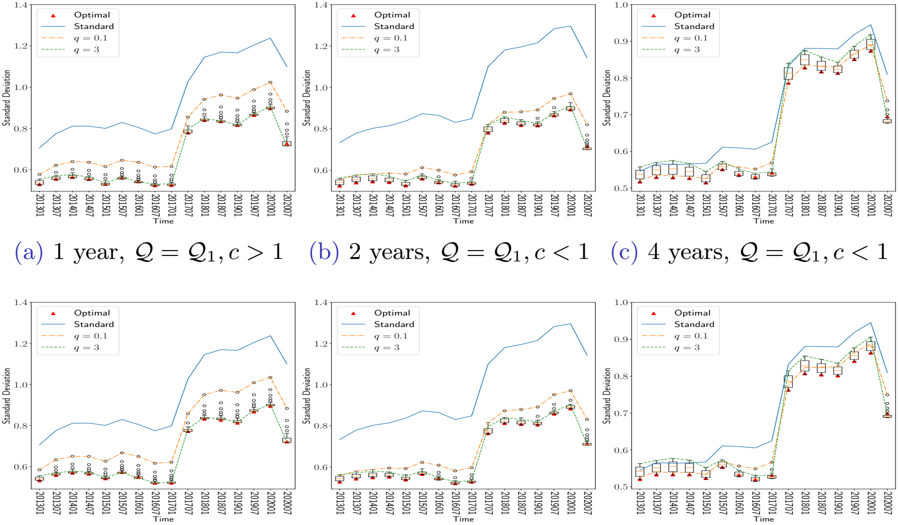


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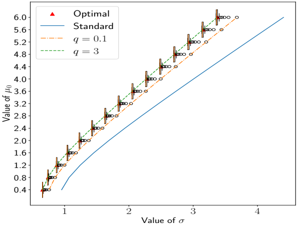
Figure 8: Standard deviation (volatility) of global minimum variance

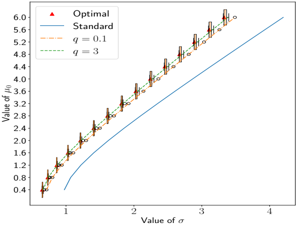
Real Data: Efficient Frontier

1. Set $\mathbf{Q} \in \mathcal{Q}_1$ or $\mathbf{Q} \in \mathcal{Q}_2$ as before. We vary μ_0 from 0.4 to 6 with an increment of 0.4. For each μ_0, carry out 2 to 5 below.
2. We build portfolio assuming no risk-free asset.
3. Let $\mathbf{r}$ be the average return vector in the testing month. The optimal portfolio is given by $\mathbf{w}^* = \mathbf{g} + \mu_0 \mathbf{h}$. We run experiments for all q values in the candidate sets, the case of $q = 0$ and the optimized q^*, which is obtained by minimizing $\frac{(\mathbf{g}+\mu_0\mathbf{h})^\top \widehat{\mathbf{\Sigma}}(\mathbf{g}+\mu_0\mathbf{h})}{(1-c/p \cdot \mathrm{tr}\widehat{\mathbf{\Sigma}}(\widehat{\mathbf{\Sigma}}+q\mathbf{I})^{-1})^2}$ over all q's.
4. We monthly roll the procedure from Jan 2013 to Jun 2023 and collect daily portfolio returns for each q value.
5. We calculate the standard deviation of the daily returns for each q value, including $q = 0$ and $q = q^*$, over the ten-year period.

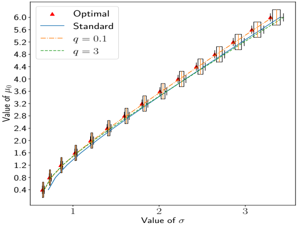
Real Data: Efficient Frontier



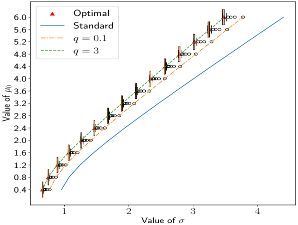
(a) 1 year, $\mathcal{Q} = \mathcal{Q}_1$



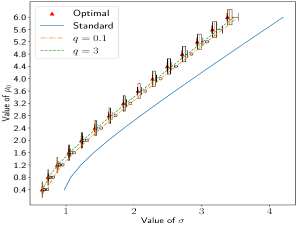
(b) 2 years, $\mathcal{Q} = \mathcal{Q}_1$



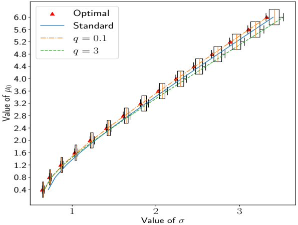
(c) 4 years, $\mathcal{Q} = \mathcal{Q}_1$



(d) 1 year, $\mathcal{Q} = \mathcal{Q}_2$



(e) 2 years, $\mathcal{Q} = \mathcal{Q}_2$



(f) 4 years, $\mathcal{Q} = \mathcal{Q}_2$

Figure 9: The corrected efficient frontier. The x-axis represents the value of σ, while the

Conclusion

Concluding Remarks

- ★ Introduced a novel in-sample approach to estimate the out-of-sample Sharpe ratio in high-dimensional portfolio optimization.
- ★ Relaxed conditions allowing arbitrary diverging spikes when $c < 1$ and K diverging spikes when $c \geq 1$.
- ★ Extended to the estimation of efficient frontier when no risk-free asset.
- ★ Used the OOS Sharpe estimator as objective to optimize the Ridge tuning parameter cycle by cycle.
- ★ Verified the performance of the estimator via extensive numerical experiments.

Thank you!